Calculation of Blackbody Radiance

What is a Blackbody?

A *blackbody* is a hypothetical object that absorbs all incident electromagnetic radiation while maintaining thermal equilibrium. No light is reflected from or passes through a blackbody, but radiation is emitted, and is called *blackbody radiation*. The prefix black is used because at room temperature such an object would emit almost no visible light, appearing black to an observer.

No physical object exactly fits this definition, but most behave at least in part as blackbodies. Calculation of the radiometric quantities associated with blackbody radiation is extremely important in physics, chemistry, optics, engineering, astronomy and many other areas.

History of blackbody theory

In 1900, Max Planck developed the modern theory describing the radiation field of a blackbody. At the time, there were two distinct models for blackbody radiation: the Rayleigh-Jeans law, which fit the measurements well at low frequencies, and Wien's law, which worked well at high frequencies, but neither worked everywhere. Planck, by making the ingenious assumption that the energy of the modes of the electromagnetic field must be quantized, developed the theory that fits observations at all parts of the spectrum. This leap marked the birth of quantum mechanics and modern physics.

Radiometric systems of units

There are many choices of units when dealing with radiometric quantities, and each discipline has its preferred units. Spectroscopists traditionally prefer wavenumber, infrared engineers use wavelength, and physicists typically deal with frequency. Thermal calculations generally involve radiated/received power, but many systems, including the human eye, operate as efficient quantum detectors, and photon flux is the appropriate measure. The choice of units is not trivial, as the functional forms differ. For example, the power emitted per unit area of a blackbody at temperature T is proportional to T^4, but the photon flux is proportional to T^3.

References containing the basic formulas abound, but it is difficult to find any single source with formulas given in each system of units. Here we collect a comprehensive set of radiometric formulas in all the common units. We consider spectral units of frequency (Hz), wavelength (µm) and wavenumber (cm^{-1}). For each, we derive the basic blackbody formulas in terms of both power (W) and photon flux. Beginning with the Planck blackbody function in units of $\text{W m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1}$, all other functions are derived. We also derive useful formulas for computing integrated band radiance, and present sample C++ computer codes in Appendix A. Appendix B describes the Doppler effect on the observed blackbody radiation spectrum of moving sources. Finally, all significant formulae are summarized in Appendix C for quick reference.

The Planck Blackbody Formula in Units of Frequency

It can be shown[1] that the power emitted per unit projected area of a blackbody at temperature T, into a unit solid angle, in frequency interval ν to $\nu + d\nu$, is

$$L_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1} \tag{1}$$

where h is Planck's constant (6.6260693×10^{-34} W s^2),
c is the speed of light (2.99792458×10^{8} m s^{-1}) and
k is Boltzmann's constant (1.380658×10^{-23} J K^{-1}).

This is the *Planck blackbody formula* (in one of many forms). The quantity L_ν is referred to as the *spectral radiance*. The frequency of the maximum spectral radiance is found by setting the derivative with respect to ν equal to zero:

$$0 = \frac{dL_\nu}{d\nu} = \frac{6h\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1} - \frac{2h\nu^3}{c^2}\frac{(h/kT)e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}$$

$$0 = 3 - \frac{h\nu}{kT}\frac{e^{h\nu/kT}}{e^{h\nu/kT}-1}$$

This gives the transcendental equation $3(1-e^{-x}) = x$, where $x = h\nu/kT$. Evaluating this numerically yields $x = a_3 \approx 2.82143937212$, so

$$\nu_{peak} = \frac{a_3 k}{h}T \quad \text{Hz} \quad . \tag{2}$$

(We use the subscript 3 to refer to the coefficient in the transcendental equation, other versions of which we will encounter). Substituting this in (1) gives

$$L_{\nu,peak} = \frac{2h\left(a_3kT/h\right)^3}{c^2}\frac{1}{e^{\frac{h(a_3kT/h)}{kT}}-1}$$

$$L_{\nu,peak} = \left(\frac{2a_3^{\,3}k^3}{h^2c^2}\frac{1}{e^{a_3}-1}\right)T^3 \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1} \tag{3}$$

Many devices and systems respond in proportion to the number of incident photons, and it is useful to express radiometric quantities in terms of photons per second rather than watts. Dividing the spectral radiance L_ν (Eq. 1) by the energy of a photon, $h\nu$, gives the spectral photon radiance

[1] c.f. "Radiometry and the Detection of Optical Radiation," by Robert W. Boyd, Wiley and Sons, 1983

$$L_\nu^P = \frac{2\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1} \quad . \tag{4}$$

The peak of L_ν^P occurs when

$$0 = \frac{dL_\nu^P}{d\nu} = \frac{4\nu}{c^2}\frac{1}{e^{h\nu/kT}-1} - \frac{2\nu^2}{c^2}\frac{(h/kT)e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}$$

$$0 = 2 - \frac{h\nu}{kT}\frac{e^{h\nu/kT}}{e^{h\nu/kT}-1}$$

This gives the transcendental equation $2(1 - e^{-x}) = x$, where $x = h\nu/kT$, with solution $x = a_2 \approx 1.59362426004$. The peak spectral photon radiance thus occurs at frequency

$$\nu_{peak}^P = \frac{a_2 k}{h}T \quad \text{Hz} \quad . \tag{5}$$

Note that the peak of L_ν^P occurs at a lower frequency than L_ν. The peak value of L_ν^P is found by substituting (5) into (4):

$$L_{\nu,peak}^P = \frac{2\left(a_2 kT/h\right)^2}{c^2}\frac{1}{e^{\frac{h(a_2 kT/h)}{kT}}-1}$$

$$L_{\nu,peak}^P = \frac{2a_2^2k^2}{h^2c^2}\frac{1}{e^{a_2}-1}T^2 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1} \tag{6}$$

Both L_ν and L_ν^P are shown in Fig 1, for several temperatures.

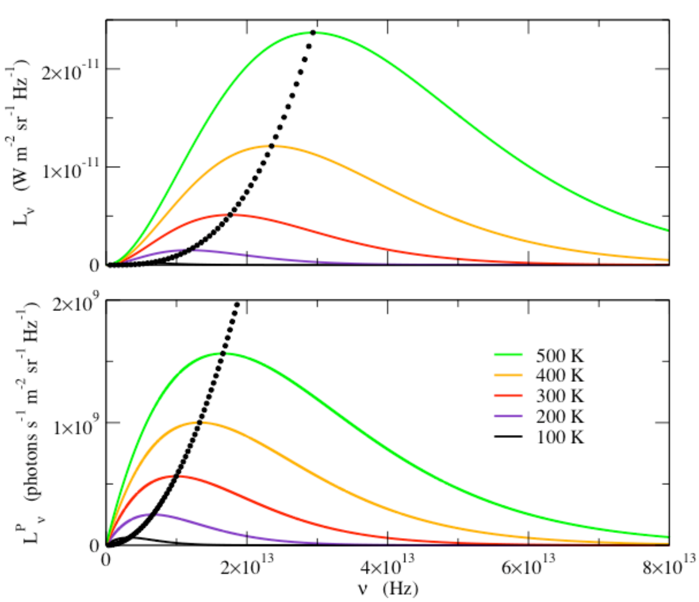

Fig 1—Spectral radiance, L_ν, (*top*) and the spectral photon radiance, L_ν^P, (*bottom*) as a function of frequency, ν, for various temperatures. The small black dots indicate the frequency and value of the peak, at 10 K temperature intervals. Note that L_ν and L_ν^P have different frequency dependences. Although the peak frequency is proportional to T for both quantities, L_ν peaks at a higher frequency than L_ν^P. Furthermore, the peak value of L_ν increases as T^3, whereas the peak value of L_ν^P increases as T^2.

Units of Wavelength

For many applications, particularly when dealing in the infrared region of the spectrum, the preferred spectral unit is wavelength in μm, $\lambda = 10^6\, c/\nu$. We can deduce the spectral radiance per μm, L_λ, from (1) by noting that

$$L_\lambda \left| d\lambda \right| = L_\nu \left| d\nu \right| \qquad\qquad L_\lambda = \left| \frac{d\nu}{d\lambda} \right| L_\nu = \frac{10^6 c}{\lambda^2} L_\nu$$

With this, and substituting $\nu = 10^6\, c/\lambda$ into (1), the spectral radiance per μm is:

$$L_\lambda = \frac{2 \times 10^{24} hc^2}{\lambda^5} \frac{1}{e^{10^6 hc/\lambda kT} - 1} \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1} \tag{7}$$

To find the wavelength of the peak, we set the derivative to zero:

$$0 = \frac{dL_\lambda}{d\lambda} = \frac{-10^{25} hc^2}{\lambda^6} \frac{1}{e^{10^6 hc/\lambda kT} - 1} + \frac{2 \times 10^{24} hc^2}{\lambda^5} \frac{\left(10^6 hc/\lambda^2 kT\right) e^{10^6 hc/\lambda kT}}{\left(e^{10^6 hc/\lambda kT} - 1\right)^2}$$

$$0 = 5 - \frac{10^6 hc}{\lambda kT} \frac{e^{10^6 hc/\lambda kT}}{e^{10^6 hc/\lambda kT} - 1}$$

Letting $x = 10^6 hc/\lambda kT$, we arrive at the transcendental equation $5(1 - e^{-x}) = x$, whose numerical solution, $x = a_5 \approx 4.96511423174$ provides

$$\lambda_{peak} = \frac{10^6 hc}{a_5 kT} \quad \mu\text{m} \tag{8}$$

The peak value, found by substituting (8) into (7), is

$$L_{\lambda,peak} = \frac{2 \times 10^{24} hc^2}{\left(\frac{10^6 hc}{a_5 kT}\right)^5} \frac{1}{e^{10^6 hc \big/ \left(\frac{10^6 hc}{a_5 kT}\right) kT} - 1}$$

$$L_{\lambda,peak} = \frac{2 a_5^{\,5} k^5}{10^6 h^4 c^3} \frac{1}{e^{a_5} - 1} T^5 \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1} \tag{9}$$

As we did above with spectral units of Hz, we can derive these radiometric quantities in terms of photons per second. Dividing (7) by the energy of a photon, $10^6\, hc/\lambda$, gives the spectral photon radiance,

$$L_\lambda^P = \frac{2 \times 10^{18} c}{\lambda^4} \frac{1}{e^{10^6 hc/\lambda kT} - 1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1}\ . \tag{10}$$

The wavelength where this peaks is found by differentiating:

$$0 = \frac{dL_\lambda^P}{d\lambda} = \frac{-8\times10^{18}c}{\lambda^5}\frac{1}{e^{10^6 hc/\lambda kT}-1} + \frac{2\times10^{18}c}{\lambda^4}\frac{\left(10^6 hc/\lambda^2 kT\right)e^{10^6 hc/\lambda kT}}{\left(e^{10^6 hc/\lambda kT}-1\right)^2}$$

$$0 = 1 - \frac{10^6 hc}{4\lambda kT}\frac{e^{10^6 hc/\lambda kT}}{e^{10^6 hc/\lambda kT}-1} \qquad \text{let} \quad x = 10^6 hc/\lambda kT$$

$$4\left(1-e^{-x}\right) = x$$

$$x = a_4 \approx 3.92069039487$$

$$\lambda_{peak}^P = \frac{10^6 hc}{a_4 kT} \qquad \mu\text{m} \quad . \tag{11}$$

The peak spectral photon radiance is

$$L_{\lambda,peak}^P = \frac{2\times10^{18}c}{\left(10^6 hc/a_4 kT\right)^4}\frac{1}{e^{\frac{10^6 hc}{\left(10^6 hc/a_4 kT\right)kT}}-1}$$

$$L_{\lambda,peak}^P = \frac{2a_4{}^4k^4}{10^6 h^4c^3}\frac{1}{e^{a_4}-1}T^4 \qquad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1} . \tag{12}$$

L_λ and $L_\lambda{}^P$ are shown in Fig 2. Note that as with units of Hz, the spectral radiance and spectral photon radiance have different behaviors, and distinctly different temperature dependences.

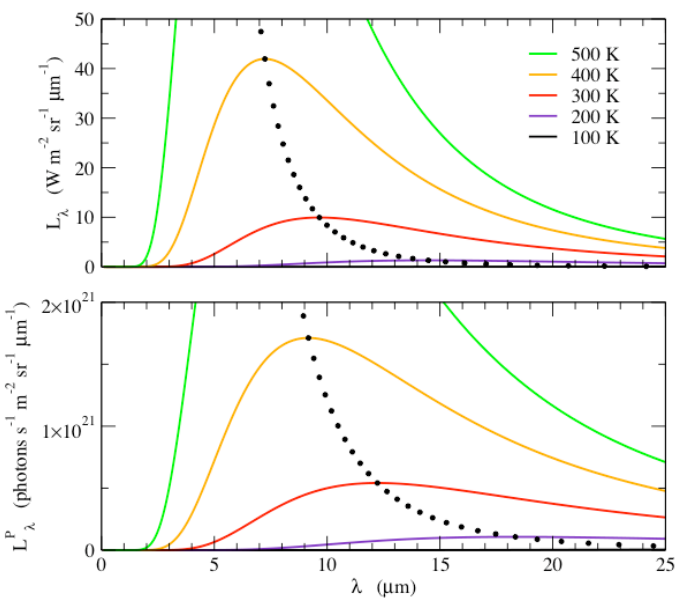

Fig 2—Spectral radiance, L_λ, (*top*) and the spectral photon radiance, $L_\lambda{}^P$, (*bottom*) as a function of wavelength, λ, for various temperatures. The small black dots indicate the wavelength and value of the peak, at 10 K temperature intervals. Note that L_λ and $L_\lambda{}^P$ have different wavelength dependences. Although the peak wavelength is inversely proportional to T for both quantities, $L_\lambda{}^P$ peaks at a longer wavelength than L_λ. Furthermore, the peak value of L_λ increases as T^5, whereas the peak value of $L_\lambda{}^P$ increases as T^4.

Units of Wavenumbers

Yet a third spectral unit, commonly used in spectroscopy, is wavenumber, the number of waves per cm: $\sigma = \nu/100c\ \text{cm}^{-1}$. Converting (1) to these units gives

$$L_\sigma = \left|\frac{d\nu}{d\sigma}\right| L_\nu = (100c) L_\nu \ .$$

$$L_\sigma = 2\times 10^8 hc^2\sigma^3 \frac{1}{e^{100hc\sigma/kT}-1} \quad \text{W m}^{-2}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1} \qquad (13)$$

Again, the peak is where the derivative with respect to wavenumber vanishes:

$$0 = \frac{dL_\sigma}{d\sigma} = 6\times 10^8 hc^2\sigma^2 \frac{1}{e^{100hc\sigma/kT}-1} - 2\times 10^8 hc^2\sigma^3 \frac{(100hc/kT)e^{100hc\sigma/kT}}{\left(e^{100hc\sigma/kT}-1\right)^2}$$

$$0 = 3 - \frac{100hc\sigma}{kT}\frac{e^{100hc\sigma/kT}}{e^{100hc\sigma/kT}-1} \qquad \text{let} \quad x = \frac{100hc\sigma}{kT}$$

$$3\left(1-e^{-x}\right) = x$$

$$x = a_3 \approx 2.82143937212$$

so

$$\sigma_{peak} = \frac{a_3 kT}{100hc} \quad \text{cm}^{-1} \quad . \qquad (14)$$

The peak value is

$$L_{\sigma,peak} = 2\times 10^8 hc^2 \left(\frac{a_3 kT}{100hc}\right)^3 \frac{1}{e^{100hc\left(\frac{a_3 kT}{100hc}\right)/kT}-1}$$

$$L_{\sigma,peak} = \frac{200 a_3^{\,3} k^3}{h^2 c}\frac{1}{e^{a_3}-1} T^3 \qquad \text{W m}^{-2}\ \text{sr}^{-1}\ \left(\text{cm}^{-1}\right)^{-1} \ . \qquad (15)$$

The spectral photon radiance is found by dividing L_σ by the energy of a photon, $100hc\sigma$:

$$L_\sigma^P = \frac{L_\sigma}{100hc\sigma} = 2\times 10^6 c\sigma^2 \frac{1}{e^{100hc\sigma/kT}-1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1}. \qquad (16)$$

We next find the wavenumber at the peak of the spectral photon radiance:

$$0 = \frac{dL_\sigma^P}{d\sigma} = 4\times10^6 c\sigma \frac{1}{e^{100hc\sigma/kT}-1} - 2\times10^6 c\sigma^2 \frac{\left(100hc/kT\right)e^{100hc\sigma/kT}}{\left(e^{100hc\sigma/kT}-1\right)^2}$$

$$0 = 2 - \frac{100hc\sigma}{kT}\frac{e^{100hc\sigma/kT}}{e^{100hc\sigma/kT}-1} \qquad \text{let } x = \frac{100hc\sigma}{kT}$$

$$2\left(1-e^{-x}\right) = x \qquad x = a_2 \approx 1.59362426004$$

and

$$\sigma_{peak}^P = \frac{a_2 k}{100hc} T \quad \text{cm}^{-1}. \tag{17}$$

The peak spectral photon radiance is

$$L_{\sigma,peak}^P = 2\times10^6 c\left(\frac{a_2 kT}{100hc}\right)^2 \frac{1}{e^{\frac{100hc\left(a_2 kT/100hc\right)}{kT}}-1}$$

$$L_{\sigma,peak}^P = 200\frac{a_2^2k^2}{h^2c}\frac{1}{e^{a_3}-1}T^2 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1}. \tag{18}$$

Fig 3 shows plots of L_σ and L_σ^P for various temperatures. Note again the important difference between the spectral radiance and spectral photon radiance.

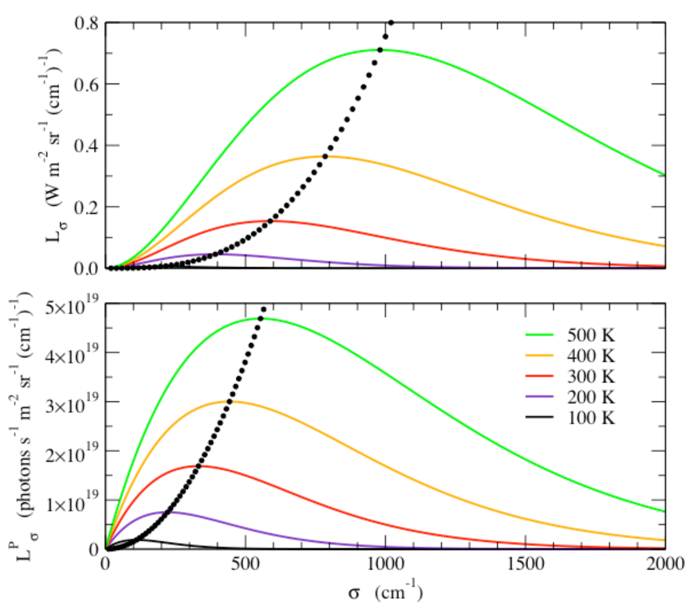

Fig 3—Spectral radiance, L_σ, (*top*) and the spectral photon radiance, L_σ^P, (*bottom*) as a function of wavenumber, σ, for various temperatures. The small black dots indicate the wavenumber and value of the peak, at 10 K temperature intervals. Note that L_σ and L_σ^P have different wavenumber dependences. Although the peak wavenumber is proportional to T for both quantities, L_σ peaks at a higher wavenumber than L_σ^P. Furthermore, the peak value of L_σ increases as T^3, whereas the peak value of L_σ^P increases as T^2.

Radiance: Integrating the Planck Equation

Above we considered three different spectral units: frequency, ν, (Hz), wavelength, λ, (μm) and wavenumber, σ, (cm^{-1}). We derived expressions for the spectral radiances L_ν, L_λ, and L_σ. To find the radiance, L (W m^{-2} sr^{-1}), we can integrate any of these over the respective spectral variable. That is,

$$L = \int_0^\infty L_\nu d\nu = \int_0^\infty L_\lambda d\lambda = \int_0^\infty L_\sigma d\sigma$$

We will perform the integration of L_ν over all frequencies, ν:

$$\begin{aligned}
L &= \int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}d\nu \\
&= 2\frac{k^3T^3}{h^2c^2}\int_0^\infty \left(\frac{h\nu}{kT}\right)^3 \frac{1}{e^{h\nu/kT}-1}d\nu \qquad \text{let} \quad x=\frac{h\nu}{kT} \qquad dx=\frac{h}{kT}d\nu \\
&= 2\frac{k^4T^4}{h^3c^2}\int_0^\infty \frac{x^3}{e^x-1}dx = 2\frac{k^4T^4}{h^3c^2}\int_0^\infty \frac{x^3e^{-x}}{1-e^{-x}}dx \qquad \text{note that} \quad \frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^\infty e^{-nx} \\
&= 2\frac{k^4T^4}{h^3c^2}\sum_{n=1}^\infty \int_0^\infty x^3e^{-nx}dx \qquad \text{integrate by parts...} \\
&= 2\frac{k^4T^4}{h^3c^2}\sum_{n=1}^\infty \frac{6}{n^4} = 12\frac{k^4T^4}{h^3c^2}\zeta(4) = 2\frac{k^4T^4}{h^3c^2}\frac{\pi^4}{15}
\end{aligned}$$

$$L = \frac{2\pi^4k^4}{15h^3c^2}T^4 \quad \text{W m}^{-2}\ \text{sr}^{-1} \quad . \tag{19}$$

(Here we used the result $\Sigma n^{-4} = \zeta(4) = \pi^4/90$, where ζ is the Reimann zeta function) The total radiated power per unit area, called the *radiant emittance* or *radiant exitance*, *M*, can be found by further integrating with respect to solid angle over the hemisphere into which the surface radiates. A source whose radiance is independent of angle is called *Lambertian* (from Lambert's cosine law of reflection). This is implicitly assumed for an ideal blackbody, and is a good approximation for many real sources. For a Lambertian source, *M* is related to *L* by

$$M = \int_0^{2\pi} d\phi \int_0^{\pi/2} L\cos\theta\,\sin\theta d\theta = 2\pi L\int_0^1 x\,dx = \pi L$$

so

$$M = \left(\frac{2\pi^5k^4}{15h^3c^2}\right)T^4 \quad \text{W m}^{-2} \quad . \tag{20}$$

This is the Stefan-Boltzmann law, and the quantity in parentheses is the Stefan-Boltzmann constant. A common mistake in deriving this result is to assume the factor is 2π rather than π, because there are 2π steradians in the hemisphere, but this neglects the cosθ reduction from Lambert's cosine law.

Similarly, the total photon radiance is found by integrating L_σ^P over all frequencies:

$$\begin{aligned}
L^P &= \int_0^\infty \frac{2\nu^2}{c^2} \frac{1}{e^{h\nu/kT}-1} d\nu \\
&= 2\frac{k^2T^2}{h^2c^2} \int_0^\infty \left(\frac{h\nu}{kT}\right)^2 \frac{1}{e^{h\nu/kT}-1} d\nu \qquad \text{let } x = \frac{h\nu}{kT} \quad dx = \frac{h}{kT} d\nu \\
&= 2\frac{k^3T^3}{h^3c^2} \int_0^\infty \frac{x^2}{e^x - 1} dx = 2\frac{k^3T^3}{h^3c^2} \int_0^\infty \frac{x^2 e^{-x}}{1-e^{-x}} dx \qquad \text{note that } \frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^\infty e^{-nx} \\
&= 2\frac{k^3T^3}{h^3c^2} \sum_{n=1}^\infty \int_0^\infty x^2 e^{-nx} dx \qquad \text{integrate by parts...} \\
&= 2\frac{k^3T^3}{h^3c^2} \sum_{n=1}^\infty \frac{2}{n^3}
\end{aligned}$$

$$L^P = \frac{4\zeta(3)k^3}{h^3c^2} T^3 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1} \tag{21}$$

Recall ζ is the Riemann zeta function. ζ(3) ≈ 1.202056903159594 is also known as Apéry's constant. Integrating L_σ^P over the hemisphere (again assuming a Lambertian source) gives

$$M^P = \frac{4\pi\zeta(3)k^3}{h^3c^2} T^3 \quad \text{photon s}^{-1}\ \text{m}^{-2} \tag{22}$$

This is the Stefan Boltzmann law for photon radiant emittance. Notice that the total photon flux is proportional to T^3, whereas the total power, given by (5), is proportional to T^4.

In-band Radiance: Integrating the Planck Equation Over a Finite Range

Above we analytically integrated the spectral radiance over the entire spectral range. The result, Eq. (20), is the well-known Stefan-Boltzmann law. Similarly, Eq. (22) gives the integrated photon radiance. As useful as the Stefan-Boltzmann law is, for many applications a finite spectral range is needed. To facilitate this, we compute the one-sided integral of the spectral radiance. We follow the method described by Widger and Woodall[2], using units of wavenumber. Note that using other spectral units produces the same result, because it represents the same physical quantity.

$$B(\sigma) = \int_{\sigma}^{\infty} L_{\sigma'} d\sigma' = \int_{\sigma}^{\infty} 2\times10^{8} hc^{2}\sigma'^{3} \frac{1}{e^{100hc\sigma'/kT}-1} d\sigma'$$

$$B(\sigma) = \int_{\sigma}^{\infty} 2\times10^{8} hc^{2}\sigma'^{3} \frac{1}{e^{100hc\sigma'/kT}-1} d\sigma' \qquad \text{let } x' = \frac{100hc\sigma'}{kT}, \; dx' = \frac{100hc}{kT} d\sigma'$$

$$B(x) = \int_{x}^{\infty} 2\times10^{8} hc^{2} \left(\frac{kT}{100hc}\right)^{3} \frac{x'^{3}}{e^{x'}-1} \frac{kT}{100hc} dx' \qquad \text{where} \quad x = \frac{100hc\sigma}{kT}$$

$$B(x) = 2\frac{k^{4}T^{4}}{h^{3}c^{2}} \int_{x}^{\infty} \frac{x'^{3}}{e^{x'}-1} dx'$$

Noting that $\frac{1}{e^{x'}-1} = \sum_{n=1}^{\infty} e^{-nx'}$, we get $B(x) = 2\frac{k^{4}T^{4}}{h^{3}c^{2}} \sum_{n=1}^{\infty} \int_{x}^{\infty} x'^{3} e^{-nx'} dx'$.

The remaining integral can be integrated by parts[3]:

$$\int_{x}^{\infty} x'^{3} e^{-nx} dx' = \left(\frac{x^{3}}{n} + \frac{3x^{2}}{n^{2}} + \frac{6x}{n^{3}} + \frac{6}{n^{4}}\right) e^{-nx}$$

This gives

$$\int_{\sigma}^{\infty} L_{\sigma'}\, d\sigma' = 2\frac{k^{4}T^{4}}{h^{3}c^{2}} \sum_{n=1}^{\infty} \left(\frac{x^{3}}{n} + \frac{3x^{2}}{n^{2}} + \frac{6x}{n^{3}} + \frac{6}{n^{4}}\right) e^{-nx} \quad \text{W m}^{-2}\ \text{sr}^{-1}$$
$$\text{where} \quad x = \frac{100hc\sigma}{kT} \tag{23}$$

Testing shows that carrying the summation up to $n = min(2+20/x, 512)$ provides convergence to at least 10 digits.

---

[2] Widger, W. K. and Woodall, M. P., Integration of the Planck blackbody radiation function, Bulletin of the Am. Meteorological Society, 57, 10, 1217-1219, Oct. 1976

[3] CRC Handbook of Chemisry and Physics, 56th edition #521

Any finite range can be computed using two one-sided integrals:

$$\int_{\sigma_1}^{\sigma_2} L_{\sigma'} d\sigma' = B(\sigma_1) - B(\sigma_2)$$

Further, the complimentary integral is easily evaluated using (19):

$$\int_{0}^{\sigma} L_{\sigma'} d\sigma' = \int_{0}^{\infty} L_{\sigma'} d\sigma' - \int_{\sigma}^{\infty} L_{\sigma'} d\sigma' = \frac{2\pi^4}{15}\frac{k^4}{h^3c^2}T^4 - B(\sigma)$$

A similar formula can be derived for the in-band photon radiance:

$$B^P(\sigma) = \int_{\sigma}^{\infty} L^P_{\sigma'} d\sigma' = \int_{\sigma}^{\infty} 2\times 10^6 c\sigma'^2 \frac{1}{e^{100hc\sigma'/kT} - 1} d\sigma' \qquad \text{let } x' = \frac{100hc\sigma'}{kT},\; dx' = \frac{100hc}{kT} d\sigma'$$

$$B^P(x) = \int_{x}^{\infty} 2\times 10^6 c\left(\frac{kT}{100hc}\right)^2 \frac{x'^3}{e^{x'} - 1}\frac{kT}{100hc} dx' \qquad \text{where } \; x = \frac{100hc\sigma}{kT}$$

$$= 2\frac{k^3T^3}{h^3c^2}\int_{x}^{\infty}\frac{x'^2}{e^{x'} - 1} dx'$$

Again using $\frac{1}{e^{x'} - 1} = \sum_{n=1}^{\infty} e^{-nx'}$, we get $B(x) = 2\frac{k^3T^3}{h^3c^2}\sum_{n=1}^{\infty}\int_{x}^{\infty} x'^2 e^{-nx'} dx'$.

Integrating by parts $\int_{x}^{\infty} x'^2 e^{-nx} dx' = \left(\frac{x^2}{n} + \frac{2x}{n^2} + \frac{2}{n^3}\right)e^{-nx}$ so

$$\int_{\sigma}^{\infty} L^P_{\sigma'} d\sigma' = 2\frac{k^3T^3}{h^3c^2}\sum_{n=1}^{\infty}\left(\frac{x^2}{n} + \frac{2x}{n^2} + \frac{2}{n^3}\right)e^{-nx} \qquad \text{photon s}^{-1}\text{ m}^{-2}\text{ sr}^{-1} \tag{24}$$

Equations (23) and (24) provide efficient formulas for computing in-band radiance. Example C++ computer source code is provided in Appendix A.

Appendix A: Algorithms for Computing In-band Radiance

Below are C++ functions for computing the integrated spectral radiance (W m^{-2} sr^{-1}) and integrated spectral photon radiance (photon s^{-1} m^{-2} sr^{-1}). The functions compute the integral from the specified wavenumber to infinity for a blackbody at the input temperature. Finite spectral regions can be computed by using this function twice-once with each end point of the spectral region. The difference of the two gives the radiance for the spectral region.

```
#include  <math.h> // for “exp” function

double planck_integral (double sigma, double temperature) {

//  integral of spectral radiance from sigma (cm-1) to infinity.
//  result is W/m2/sr.
//  follows Widger and Woodall, Bulletin of the American Meteorological
//  Society, Vol. 57, No. 10, pp. 1217

//  constants
      double Planck =  6.6260693e-34 ;
      double  Boltzmann = 1.380658e-23 ;
      double  Speed_of_light = 299792458.0 ;
      double  Speed_of_light_sq = Speed_of_light * Speed_of_light ;

//  compute powers of x, the dimensionless spectral coordinate
      double c1 =  (Planck*Speed_of_light/Boltzmann) ;
      double x =  c1 * 100 * sigma / temperature ;
      double x2 = x *  x  ;
      double x3 = x *  x2 ;

//  decide how many terms of sum are needed
      double iterations = 2.0 + 20.0/x ;
      iterations = (iterations<512) ? iterations : 512 ;
      int iter = int(iterations) ;

//  add up terms of sum
      double sum = 0  ;
      for (int n=1;  n<iter; n++) {
            double  dn = 1.0/n ;
            sum  += exp(-n*x)*(x3 + (3.0 * x2 + 6.0*(x+dn)*dn)*dn)*dn;
            }

//  return result, in units of W/m2/sr
      double c2 =  (2.0*Planck*Speed_of_light_sq) ;
      return  c2*pow(temperature/c1,4)*sum ;

}
```

```
#include  <math.h>  // for "exp"  function

double planck_photon_integral (double sigma, double temperature) {

//  integral of spectral photon radiance from sigma (cm-1) to infinity.
//  result is photons/s/m2/sr.
//  follows Widger and Woodall, Bulletin of the American Meteorological
//  Society, Vol. 57, No. 10, pp. 1217

//  constants
      double Planck =  6.6260693e-34 ;
      double Boltzmann = 1.380658e-23 ;
      double Speed_of_light = 299792458.0 ;

//  compute powers of x, the dimensionless spectral coordinate
      double c1 =  Planck*Speed_of_light/Boltzmann ;
      double x =  c1*100*sigma/temperature ;
      double x2 = x *  x  ;

//  decide how many terms of sum are needed
      double iterations = 2.0 + 20.0/x ;
      iterations = (iterations<512) ? iterations : 512 ;
      int iter = int(iterations) ;

//  add up terms of sum
      double sum = 0  ;
      for (int n=1;  n<iter; n++) {
            double  dn = 1.0/n ;
            sum  += exp(-n*x) * (x2 + 2.0*(x + dn)*dn)*dn ;
            }

//  return result, in units of photons/s/m2/sr
      double kTohc =  Boltzmann*temperature/(Planck*Speed_of_light) ;
      double c2 =  2.0* pow(kTohc,3)*Speed_of_light ;
      return c2 *sum  ;

}
```

Appendix B: The Doppler Effect

The observed frequency ν' of light emitted from a source moving with velocity u as depicted in Fig. B1 is given by [4]

$$\nu' = \frac{\nu}{\sqrt{1-u^2/c^2}}\left(1-\frac{u}{c}\cos\theta\right) \quad , \tag{B1}$$

where ν is the frequency of the light in the rest frame of the source, and c is the speed of light, 299,792 km/s.

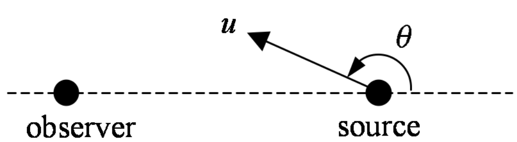

Fig B1—Geometry of a moving source. If the source is receding, the measured frequency will be decreased (“red shifted”), compared to the frequency in the rest frame of the source. If the source is approaching, the measured frequency will be increased (“blue shifted”). Eq. B1 gives the relationship between the frequencies in the two reference frames.

If the velocity is purely radial, then Eq. B1 reduces to

$$\nu' = \nu\sqrt{\frac{c-u}{c+u}} = \beta\nu \quad , \tag{B2}$$

where $\beta^2 = (c-u)/(c+u)$. Note that in Eq. B2 we have adopted the convention that $u > 0$ indicates a receding source. For our application, we assume the velocity is purely radial (θ= 0° or 180°) and use Eq. 2.[5] The magnitude of the Doppler effect for some typical situations is given in Table B1.

Table B1—Examples of Doppler shifts for 1000 cm^{-1} (10 µm) light

source	*recession velocity (km/s)*	*shift (cm^{-1})*
geostationary satellite	3	0.01
low Earth orbit satellite	7	0.02
orbital speed of Earth	30	0.1
typical star	300	1
most distant galaxy	75,000	225

[4] J. D. Jackson, *Classical Electrodynamics*, 2nd ed., pp 522

[5] The tangential effect is small in most realistic cases anyway. For example, 1000 cm^{-1} light (10 µm) from a source moving at 100 km/s perpendicular to the line of sight is shifted only 56×10^{-6} cm^{-1}.

Eq. B2 gives the Lorentz transformation for a monochromatic frequency ν. However, for a continuous spectrum, we cannot simply scale all frequencies.[6] We must also apply the Lorentz transformation to the (necessarily finite) aperture collecting the radiation. An aperture subtending solid angle Ω in the rest frame of the source will appear to have solid angle

$$\Omega' = \Omega\left(\frac{c+u}{c-u}\right) = \beta^{-2}\Omega \qquad \text{(B3)}$$

when receding with velocity u. Suppose now that the source has a rest-frame radiance of $L^P(\nu)$. An aperture of size Ω in its rest frame will receive a photon flux of

$$N(\nu) = \Omega L^P(\nu)$$

If the aperture is receding with velocity u, the photon flux received from the receding source will be

$$N'(\nu) = \beta^2 \Omega L^P\left(\beta^{-1}\nu\right)$$

If the source is a blackbody at temperature T, (Eq. 4), we have

$$\begin{aligned} N'(\nu) &= \beta^2 \Omega' \frac{2h\left(\nu/\beta\right)^2}{c^2} \frac{1}{e^{h\nu/\beta kT} - 1} \\ &= \Omega' \frac{2h\nu^2}{c^2} \frac{1}{e^{h\nu/k(\beta T)} - 1} \end{aligned}$$

If we interpret this radiation as coming from a stationary blackbody, that is, $L'(\nu) = N'(\nu)/\Omega'$, then the effective temperature is

$$T' = T\sqrt{\frac{c-u}{c+u}} \quad , \qquad \text{(B4)}$$

Thus the spectrum of a receding blackbody appears identical to a cooler, stationary blackbody.

---

[6] T. P. Gill, "The Doppler Effect", Logos Press, Inc., 1965

Appendix C: Summary of Formulas

In the following formulas, ν is in Hz, λ in μm, σ in cm^{-1}, and T in K. The rest of the constants are given below.

$$L_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1}$$

$$\nu_{peak} = \frac{a_3 k}{h}T \quad \text{Hz} \qquad L_{\nu,peak} = \frac{2a_3{}^3k^3}{h^2c^2}\frac{1}{e^{a_3}-1}T^3 \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1}$$

$$L_\nu^P = \frac{2\nu^2}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1}$$

$$\nu_{peak}^P = \frac{a_2 k}{h}T \quad \text{Hz} \qquad L_{\nu,peak}^P = \frac{2a_2{}^2k^2}{h^2c^2}\frac{1}{e^{a_2}-1}T^2 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \text{Hz}^{-1}$$

$$L_\lambda = \frac{2\times10^{24}hc^2}{\lambda^5}\frac{1}{e^{10^6 hc/\lambda kT}-1} \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1}$$

$$\lambda_{peak} = \frac{10^6 hc}{a_5 kT} \quad \mu\text{m} \qquad L_{\lambda,peak} = \frac{2a_5{}^5k^5}{10^6h^4c^3}\frac{1}{e^{a_5}-1}T^5 \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1}$$

$$L_\lambda^P = \frac{2\times10^{18}c}{\lambda^4}\frac{1}{e^{10^6 hc/\lambda kT}-1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1}$$

$$\lambda_{peak}^P = \frac{10^6 hc}{a_4 kT} \quad \mu\text{m} \qquad L_{\lambda,peak}^P = \frac{2a_4{}^4k^4}{10^6h^4c^3}\frac{1}{e^{a_4}-1}T^4 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ \mu\text{m}^{-1}$$

$$L_\sigma = 2\times10^8 hc^2\sigma^3\frac{1}{e^{100hc\sigma/kT}-1} \quad \text{W m}^{-1}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1}$$

$$\sigma_{peak} = \frac{a_3 k}{100hc}T \quad \text{cm}^{-1} \qquad L_{\sigma,peak} = \frac{200a_3{}^3k^3}{h^2c}\frac{1}{e^{a_3}-1}T^3 \quad \text{W m}^{-2}\ \text{sr}^{-1}\ \left(\text{cm}^{-1}\right)^{-1}$$

$$L_\sigma^P = 2\times10^6 c\sigma^2\frac{1}{e^{100hc\sigma/kT}-1} \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1}$$

$$\sigma_{peak}^P = \frac{a_2 k}{100hc}T \quad \text{cm}^{-1} \qquad L_{\sigma,peak}^P = \frac{200a_2^2k^2}{h^2c}\frac{1}{e^{a_3}-1}T^2 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}\ (\text{cm}^{-1})^{-1}$$

$$L = \frac{2\pi^4 k^4}{15h^3c^2} T^4 \quad \text{W m}^{-2}\ \text{sr}^{-1}$$
$$M = \frac{2\pi^5 k^4}{15h^3c^2} T^4 \quad \text{W m}^{-2}$$

$$L^P = \frac{4\zeta(3)k^3}{h^3c^2} T^3 \quad \text{photon s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}$$
$$M^P = \frac{4\pi\zeta(3)k^3}{h^3c^2} T^3 \quad \text{photon s}^{-1}\ \text{m}^{-2}$$

$$\int_\sigma^\infty L_{\sigma'}\, d\sigma' = 2\frac{k^4T^4}{h^3c^2}\sum_{n=1}^{\infty}\left(\frac{x^3}{n}+\frac{3x^2}{n^2}+\frac{6x}{n^3}+\frac{6}{n^4}\right)e^{-nx} \quad \text{W m}^{-2}\ \text{sr}^{-1}$$
$$\int_\sigma^\infty L^P_{\sigma'} d\sigma' = 2\frac{k^3T^3}{h^3c^2}\sum_{n=1}^{\infty}\left(\frac{x^2}{n}+\frac{2x}{n^2}+\frac{2}{n^3}\right)e^{-nx} \quad \text{photons s}^{-1}\ \text{m}^{-2}\ \text{sr}^{-1}$$
$$\text{where} \quad x = \frac{100hc\sigma}{kT}$$

$$T' = T\sqrt{\frac{c-u}{c+u}}$$ (u is relative velocity of source, $u > 0$ indicating recession)

$h = 6.6260693 \times 10^{-34}$	W s^2	Planck's constant
$c = 2.99792458 \times 10^{8}$	m / s	speed of light
$k = 1.380658 \times 10^{-23}$	J / K	Boltzmann's constant
$\zeta(3) = 1.202056903159594$		Apéry's constant, Riemann Zeta function
$a_2 = 1.59362426004$		solution to $2(1-e^{-x}) - x = 0$
$a_3 = 2.82143937212$		solution to $3(1-e^{-x}) - x = 0$
$a_4 = 3.92069039487$		solution to $4(1-e^{-x}) - x = 0$
$a_5 = 4.96511423174$		solution to $5(1-e^{-x}) - x = 0$